NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3636

THE ACCURACY OF THE SUBSTITUTE-STRINGER APPROACH FOR DETERMINING THE BENDING FREQUENCIES OF MULTISTRINGER BOX BEAMS

By William W. Davenport

Langley Aeronautical Laboratory
Langley Field, Va.

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SUMMARY

The accuracy of the substitute-stringer approach for including the effects of shear lag in the calculation of the transverse modes and frequencies of multistringer box beams is investigated. Box beams, the covers of which consist of normal-stress-carrying stringers on sheets which carry not only shear but also normal stress, are analyzed exactly. Frequencies of beams with various numbers of stringers, obtained by means of this exact analysis, serve to determine the possible accuracy of the frequencies obtained by the substitute-stringer approach.

INTRODUCTION

The use of a substitute-stringer approach for including the effects of shear lag in the vibrational analysis of built-up box beams was investigated in reference 1. Various thin-walled rectangular tubes were idealized to substitute-stringer structures and the frequencies of the first few bending modes of the idealized structures were compared with those of the original beams. The results indicated where the substitute stringer should be located in order to take into account accurately the effect of shear lag on the frequency; for a fairly wide variety of tube proportions, locating the substitute stringer midway between the web and the centroid of the half-cover yielded frequencies accurate within 1 or 2 percent for higher as well as lower modes of vibration. However, because the rectangular tubes employed in the investigation had constant wall thickness there was no definite information given regarding the accuracy of frequencies that may be obtained for a beam which has discrete flanges and stringers in the covers and different web and cover-sheet thickness.

The purpose of this report is to examine the accuracy of the substitute-stringer approach when applied to a beam more nearly representative of actual built-up box beams. Frequencies of various realistic multistringer beams of the type shown in figure 1(a) are determined by means of the substitute-stringer approach as given in reference 1. These frequencies are then compared with the frequencies obtained by means of an exact vibrational analysis of a multistringer beam. Conclusions are thereby made with regard to the proper location of the substitute stringer.

SYMBOLS

A	total cross-sectional area of stringers in a half-cover of multistringer box beam
A_F	cross-sectional area of flange of substitute-stringer structure
A_L	cross-sectional area of a substitute stringer
A_O	cross-sectional area of flange of multistringer box beam
A_p	cross-sectional area of pth stringer of multistringer box beam
A_S	effective shear-carrying area
a	half-depth of beam
b	half-width of beam
$b_O = a$	
b_p	width of pth panel
b_{r+1}	half-width of middle panel when N is even, nonzero constant when N is odd
b_S	distance between web and adjacent substitute stringer
b_C	distance between web and centroid of area of half-cover
$B_i^{(q)}$	parameter defined by equation (A17) or equation (A40)
C	constant
$C_i^{(q)}$	parameter defined by equation (A18) or equation (A41)
$D_i^{(q)}$	parameter defined by equation (A25) or equation (A42)

E modulus of elasticity

$$F_q = \sqrt{\frac{EA_{q-1}b_q}{Gt_CL^2\left(1 + \theta_N\delta_{(r+1),q}\right)}}$$

G shear modulus of elasticity (taken equal to E/2.65 herein)

$$H = \frac{t_Wb_1}{t_Ca}$$

I bending moment of inertia

$$K_{ij}^{(q)} = P_q^2\left(\frac{i\pi}{2}\right)^2\left(1 + \delta_{0j}\right) + (j\pi)^2$$

k_B frequency coefficient, $\omega\sqrt{\frac{\mu L^4}{EI}}$

k_S coefficient of shear rigidity, $\frac{1}{L}\sqrt{\frac{EI}{GA_S}}$

$$k_i = \left(\frac{i\pi}{2}\right)^2 - k_B^2k_S^2$$

L half-length of beam

N number of stringers in a cover

$N_i^{(p)}$ parameter defined by equations (A34) and (A35) or by equations (A43) and (A44)

$$P_q = \sqrt{\frac{Eb_q^2}{GL^2}}$$

$$R_i^{(q)} = F_q^2\left(\frac{i\pi}{2}\right)^2 + H\delta_{1q}$$

$$r = \frac{1}{2}\left(N + \theta_N\right)$$

$S_i^{(p)}$	determinant given by equation (A30)
t_C	cover-sheet thickness
t_C'	effective cover-sheet thickness for normal stress
t_W	web thickness
T	maximum kinetic energy
U	maximum strain energy
u_p	longitudinal displacement of a point on the pth panel
w	vertical displacement of cross section of beam
x	longitudinal coordinate
y_p	chordwise coordinate for pth panel
$a_{mn}^{(p)}, c_m$	Fourier series coefficients
i,j,p,q,m,n	integers

$$\beta_q = \frac{b_q}{2Gt_C L}$$

δ_{ij}	Kronecker delta (0 when $i \neq j$; 1 when $i = j$)

$$\theta_N = \frac{1}{2}\left[1 - (-1)^N\right]$$

$\lambda_i^{(p)}$	Lagrangian multiplier
μ	mass of beam per unit length
$\phi_i^{(p)}$	parameter defined by equation (A8)
ω	natural frequency
ω_e	natural frequency of multistringer beam in figure 3

THE MULTISTRINGER BOX BEAM AND THE SUBSTITUTE-STRINGER STRUCTURES

A doubly symmetrical multistringer box beam of the type considered in the present paper is shown in figure 1(a). In order that the covers of this structure may behave realistically with regard to shear-lag effects, the cover sheets are permitted to carry not only shear but also normal stress. The following simplifying assumptions are made:

(1) The flanges and stringers carry only normal stress.

(2) The webs carry only shear (the bending resistance of the web is included in the flanges).

(3) Longitudinal inertia (rotary inertia) is neglected.

(4) The cross sections maintain their shape.

Assumptions (2) and (3) are generally known to be good for reasonably shallow beams such as those considered in this report; assumption (4) is good if a normal amount of bulkhead stiffness is present.

In the appendix an exact vibration analysis, similar to those of references 1 and 2, is carried out for a multistringer beam of the type shown in figure 1(a). The Rayleigh-Ritz energy procedure is used in conjunction with appropriate Fourier series and Lagrangian multipliers to obtain frequency equations for a box beam with any number of stringers. Although the analysis in the appendix allows unequal spacing of the stringers, only the case of equally spaced stringers is considered hereinafter.

The substitute-stringer structure for the beam of figure 1(a) is shown in figure 1(b). The flange-web combination of this structure is the same as that of its prototype; the covers, however, consist of substitute stringers which carry only normal stress and sheets which carry only shear. The magnitude of the shear-lag effect in this structure depends on the location of the substitute stringers. A suggested stringer location given in reference 1 is

$$b_S = 0.5b_C \tag{1}$$

where b_S is the distance between a flange and the adjacent substitute stringer and b_C is the distance between a web and the centroid of normal-stress-carrying area of the half-cover.

A vibration analysis of the substitute-stringer structure is carried out in appendix B of reference 1.

COMPARISON OF FREQUENCIES AS OBTAINED BY THE MULTISTRINGER

SOLUTION AND THE SUBSTITUTE-STRINGER APPROACH

In order to investigate the possible accuracy of the substitute-stringer approach for determining the frequencies of multistringer box beams, comparisons are made between the frequencies of four multistringer box beams as obtained by the analysis given in the appendix and the corresponding frequencies of their substitute-stringer structures as obtained by the analysis given in reference 1. The first three symmetrical transverse modes of each beam with free-free end conditions are considered.

The four beams have cross sections as shown in figure 2 and are identical except for the manner in which the total stringer area is distributed. Proportions common to all four beams are

$$\frac{L}{b} = 6.0 \qquad \frac{b}{a} = 6.0 \qquad \frac{t_W}{t_C} = 1.25 \qquad \frac{A_O}{at_W} = 1.89 \qquad \frac{A}{bt_C} = 0.75$$

where L, b, a, t_W, and t_C are, respectively, the half-length, the half-width, the half-depth, the web thickness, and the cover-sheet thickness; the total cross-sectional area of the stringers in a half-cover is given by A and the cross-sectional area of a flange is given by A_O. In addition to the preceding properties, the four beams are alike in that stringers of each beam are equally spaced across the covers. The beams thus differ only in the number of stringers on a cover; the cases considered are for $N = 1, 4, 7,$ and ∞.

In figure 2 the various relations needed to obtain the parameters of the multistringer solution for the four beams are given. When N is infinite the stringers are smeared out to a "stringer sheet" for which the effective normal-stress-carrying thickness t_C' is different from the actual cover-sheet thickness t_C.

The cross section of the substitute-stringer idealization is also shown in figure 2. In accordance with the idealizing procedure of the substitute-stringer approach, the following relations are used to determine the parameters needed in the substitute-stringer solution of reference 1:

$$A_F = A_O \qquad\qquad A_L = A + bt_C$$

where A_F and A_L are, respectively, the cross-sectional area of a flange and stringer of the substitute-stringer beam. It should be noted that the only geometric property of the substitute-stringer structure which may be different for each of the four beams is the value of b_S. For the substitute-stringer location defined by equation (1), b_S varies by virtue of the variation of b_C. The values of b_C for the four beams are:

For 1 stringer,

$$b_C = 0.714b$$

For 4 stringers,

$$b_C = 0.543b$$

For 7 stringers,

$$b_C = 0.531b$$

For infinity stringers,

$$b_C = 0.500b$$

The frequency coefficients $k_B = \omega\sqrt{\frac{\mu L^4}{EI}}$ obtained from both the substitute-stringer solution where $b_S = 0.5b_C$ and the multistringer solution for each of the four beams are given in table I. In addition to these results, the frequency coefficients obtained by a solution which includes transverse shear but not shear lag have been included. This latter set has been presented in order to demonstrate the influence of shear lag and was obtained from equation (15) of reference 3 with the rotary inertia parameter k_{RI} equal to zero.

The influence of stringer location on the frequency ω of the substitute-stringer structure for each of the four beams is presented graphically in figure 3. The frequency is given in the form of its relative error $\frac{\omega}{\omega_e} - 1$ when compared with the exact frequency ω_e of the multistringer structure; the curves indicate how the errors vary with substitute-stringer location.

DISCUSSION OF RESULTS

The results given in table I show good agreement between the frequency coefficients as obtained by the substitute-stringer approach with

$b_S/b_C = 0.5$ and those obtained by the multistringer solution. However, as may be noted, the frequency coefficients for the substitute structure with $b_S/b_C = 0.5$ are slightly high for all but the third modes of the cases for $N = 1$ and $N = \infty$. Reference to figure 3 indicates that, for all but the case of $N = \infty$, the maximum errors for the modes considered would be reduced by using a value of $b_S/b_C = 0.55$; for example, for the case of $N = 7$, the maximum error for this new value would be less than 1 percent.

The results of table I indicate that the stringer area distribution has little influence on the frequency; however, the assumption should not be made that the influence of shear lag is negligible. As may be seen in table I, the reduction in frequency due to the inclusion of shear-lag effects is 4.8, 16.7, and 19.1 percent, respectively, for the first three symmetrical modes of the case where $N = 7$.

CONCLUDING REMARKS

The numerical results of the present paper indicate that the substitute-stringer approach can yield accurate frequencies for multistringer box beams. Review of the numerical results of NACA Technical Note 3158 together with those of the present paper suggests that a value of 0.55 for b_S/b_C (ratio of the distance between web and adjacent substitute stringer to distance between web and centroid of the area of the half-cover) defines a slightly more appropriate substitute-stringer location for built-up box beams than $b_S/b_C = 0.50$, the value suggested in NACA Technical Note 3158.

Langley Aeronautical Laboratory,
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APPENDIX

VIBRATION SOLUTION OF A MULTISTRINGER BOX BEAM

The simplifying assumptions for the multistringer beam are given in the body of the paper and a cross section consisting of two cases is shown in figure 4. Case I applies when N is even and case II when N is odd, where N is the number of stringers in a cover. In addition to the dimensions and coordinates defined by figure 4, the length of the beam is assigned the value $2L$, and x is defined as the longitudinal coordinate with its origin at the midpoint of the beam. It should be noted that b_{r+1}, the distance between the rth stringer and the center line for case I, is half the width of the middle panel.

For a transverse mode of vibration let $w(x)$ be the amplitude of a vertical displacement of a cross section and let $u_p(x,y_p)$ be the amplitude of a longitudinal displacement of a point on the pth panel. Then the maximum strain energy is

$$U = 2E\sum_{p=1}^{r+1}\frac{A_{p-1}}{1+\theta_N\delta_{(r+1),p}}\int_{-L}^{L}\left(\frac{\partial u_p(x,0)}{\partial x}\right)^2 dx +$$

$$2Et_C\sum_{p=1}^{r+1}\left(1-\theta_N\delta_{(r+1),p}\right)\int_{-L}^{L}\int_0^{b_p}\left(\frac{\partial u_p(x,y_p)}{\partial x}\right)^2 dy_p\,dx +$$

$$2Gt_C\sum_{p=1,2}^{r+1}\left(1-\theta_N\delta_{(r+1),p}\right)\int_{-L}^{L}\int_0^{b_p}\left(\frac{\partial u_p(x,y_p)}{\partial y_p}\right)^2 dy_p\,dx +$$

$$2Gat_W\int_{-L}^{L}\left(\frac{dw}{dx}-\frac{u_1(x,0)}{a}\right)^2 dx \qquad \text{(A1)}$$

where

$$\theta_N = \frac{1-(-1)^N}{2} \qquad \text{(A2)}$$

and

$$\delta_{(r+1),p} = 0 \qquad (p \neq r+1) \qquad \text{(A3)}$$

$$\delta_{(r+1),(r+1)} = 1$$

The maximum kinetic energy when the influence of longitudinal inertia is neglected is

$$T = \frac{1}{2} \mu\omega^2 \int_{-L}^{L} w^2 dx \tag{A4}$$

where μ is the mass per unit length of the structure which the multistringer beam represents. It should be noted that equations (A1) to (A4) hold for $N \geqq 0$.

In the following sections appropriate infinite trigonometric series are assumed to represent the displacements and then the Rayleigh-Ritz energy procedure is applied.

Symmetrical Transverse Modes of a Free-Free Multistringer Beam

Appropriate trigonometric series for the symmetrical transverse modes of a free-free beam are

$$u_p(x,y_p) = \sum_{m=1,3,5}^{\infty} \sum_{n=0,1,2}^{\infty} a_{mn}^{(p)} \sin \frac{m\pi x}{2L} \cos \frac{n\pi y_p}{b_p}$$

$$(p = 1,2, \ldots (r + 1)) \tag{A5}$$

$$w(x) = C + \sum_{m=1,3,5}^{\infty} c_m \cos \frac{m\pi x}{2L} \tag{A6}$$

The choice of these particular series was guided by the orthogonality required for simplification of the expressions in the strain-energy equation; C in equation (A6) is introduced in order to allow $w(\pm L)$ to be unrestricted.

Because of the geometry of the structure the following restraining relations must hold:

$$u_p(x,b_p) - u_{p+1}(x,0) = 0 \qquad (p = 1,2, \ldots r) \tag{A7}$$

Substituting equation (A5) into equation (A7), multiplying by $\sin \frac{i\pi x}{2L}$ where $i = 1,3,5, \ldots$, and integrating from $-L$ to L yields

$$\sum_{n=0,1,2}^{\infty} \left[a_{in}^{(p)}(-1)^n - a_{in}^{(p+1)}\right] = \phi_i^{(p)} = 0$$

$$(p = 1,2, \ldots r)(i = 1,3,5, \ldots) \tag{A8}$$

In accordance with the Rayleigh-Ritz procedure the expression

$$U - T - \sum_{p=1,2}^{r} \sum_{m=1,3,5}^{\infty} \lambda_m^{(p)} \phi_m^{(p)} \qquad \text{(A9)}$$

is held stationary with respect to the coefficients of the assumed series. The λ's in expression (A9) are Lagrangian multipliers introduced to maintain the condition stated in (A8). The expressions for U and T of equation (A9) are given by equations (A1) and (A4), respectively, and the displacements are in turn given by equations (A5) and (A6). Thus, differentiating expression (A9) with respect to the $a_i^{(q)}$'s, c_i's, and C independently and setting the respective results equal to zero yields the following equations:

$$K_{ij}^{(1)} a_{ij}^{(1)} + 2R_i^{(1)} \sum_{n=0,1,2}^{\infty} a_{in}^{(1)} + \frac{2a}{L} H\left(\frac{i\pi}{2}\right) c_i - \beta_1 \lambda_i^{(1)} (-1)^j = 0$$

$$(i = 1,3,5, \ldots)(j = 0,1,2, \ldots) \qquad \text{(A10)}$$

$$\left(1 - \theta_N \delta_{(r+1),q}\right) K_{ij}^{(q)} a_{ij}^{(q)} + 2R_i^{(q)} \sum_{n=0,1,2}^{\infty} a_{in}^{(q)} - \beta_q \left[\lambda_i^{(q)} (-1)^j \left(1 - \delta_{(r+1),q}\right) - \lambda_i^{(q-1)}\right] = 0$$

$$(i = 1,3,5, \ldots)(j = 0,1,2, \ldots)(q = 2,3, \ldots r + 1)(r \geqq 1) \qquad \text{(A11)}$$

$$k_i c_i + \frac{L}{a}\left(\frac{i\pi}{2}\right) \sum_{n=0,1,2}^{\infty} a_{in}^{(1)} - 2k_B^2 k_S^2 \left(\frac{2}{i\pi}\right)(-1)^{\frac{i-1}{2}} C = 0$$

$$(i = 1,3,5, \ldots) \qquad \text{(A12)}$$

$$k_B^2 \left[C + \sum_{m=1,3,5}^{\infty} c_m \left(\frac{2}{m\pi}\right)(-1)^{\frac{m-1}{2}}\right] = 0 \qquad \text{(A13)}$$

where

$$K_{ij}^{(q)} = P_q^2 \left(\frac{i\pi}{2}\right)^2 \left(1 + \delta_{0j}\right) + (j\pi)^2 \qquad \text{(A14a)}$$

$$P_q^{\,2} = \frac{Eb_q^{\,2}}{GL^2} \tag{A14b}$$

$$R_i^{(q)} = F_q^{\,2}\left(\frac{i\pi}{2}\right)^2 + H\delta_{1q} \tag{A14c}$$

$$F_q^{\,2} = \frac{EA_{q-1}b_q}{Gt_CL^2\left(1 + \theta_N\delta_{(r+1),q}\right)} \tag{A14d}$$

$$H = \frac{t_Wb_1}{t_Ca} \tag{A14e}$$

$$k_B^{\,2} = \frac{\mu\omega^2L^4}{EI} \tag{A14f}$$

$$k_S = \frac{EI}{GA_SL^2} \tag{A14g}$$

$$A_S = 4at_W \tag{A14h}$$

$$\beta_q = \frac{b_q}{2Gt_CL} \tag{A14i}$$

$$k_i = \left[\left(\frac{i\pi}{2}\right)^2 - k_B^{\,2}k_S^{\,2}\right] \tag{A14j}$$

Dividing equation (A10) by $K_{ij}^{(1)}$ and summing over $j = 0,1,2, \ldots$ gives, on solving for $\sum_{j=0,1,2}^{\infty} a_{ij}^{(1)}$:

$$\sum_{j=0,1,2}^{\infty} a_{ij}^{(1)} = B_i^{(1)}\beta_1\lambda_i^{(1)} - \frac{2a}{L}\,H\left(\frac{i\pi}{2}\right)C_i^{(1)}c_i$$

$$(i = 1,3,5, \ldots) \tag{A15}$$

Similarly, for equation (A11):

$$\sum_{j=0,1,2}^{\infty} a_{ij}^{(q)} = B_i^{(q)}\beta_q\lambda_i^{(q)}\left(1 - \delta_{(r+1)q}\right) - C_i^{(q)}\beta_q\lambda_i^{(q-1)}$$

$$(i = 1,3,5, \ldots)(q = 2,3, \ldots r + 1) \qquad \text{(A16)}$$

where

$$B_i^{(q)} = \frac{\displaystyle\sum_{j=0,1,2}^{\infty} \frac{(-1)^j}{K_{ij}^{(q)}}}{1 + 2R_i^{(q)} \displaystyle\sum_{j=0,1,2}^{\infty} \frac{1}{K_{ij}^{(q)}}} \qquad (q = 1,2, \ldots r) \qquad \text{(A17)}$$

and

$$C_i^{(q)} = \frac{\displaystyle\sum_{j=0,1,2}^{\infty} \frac{1}{K_{ij}^{(q)}}}{\left(1 - \theta_N\delta_{(r+1),q}\right) + 2R_i^{(q)} \displaystyle\sum_{j=0,1,2}^{\infty} \frac{1}{K_{ij}^{(q)}}}$$

$$(q = 1,2, \ldots r + 1) \qquad \text{(A18)}$$

In equation (A17) the definition of $B_i^{(r+1)}$ has been neglected because when $q = r + 1$ in equation (A16) the entire term containing $B_i^{(r+1)}$ drops out.

By substituting equations (A15) and (A16) into equations (A10) and (A11) the following equations may be obtained:

$$a_{ij}^{(1)} = \frac{1}{K_{ij}^{(1)}}\left\{c_1\left(\frac{2a}{L}\right)H\left(\frac{i\pi}{2}\right)\left[2R_i^{(1)}C_i^{(1)} - 1\right] + \beta_1\lambda_i^{(1)}\left[(-1)^j - 2R_i^{(1)}B_i^{(1)}\right]\right\}$$

$$(i = 1,3,5, \ldots)(j = 0,1,2, \ldots) \qquad \text{(A19)}$$

$$a_{ij}^{(q)} = \frac{1}{K_{ij}^{(q)}}\left\{\beta_q\lambda_i^{(q-1)}\left[2R_i^{(q)}C_i^{(q)} - 1\right] + \beta_q\lambda_i^{(q)}\left[(-1)^j - 2R_i^{(q)}B_i^{(q)}\right]\right\}$$

$$(i = 1,3,5, \ldots)(j = 0,1,2, \ldots)(q = 2,3, \ldots r) \qquad \text{(A20)}$$

Equations (A19) and (A20), when multiplied by $(-1)^j$ and summed over $j = 0,1,2, \ldots,$ yield

$$\sum_{j=0,1,2}^{\infty} (-1)^j a_{ij}^{(1)} = -B_i^{(1)}\left(\frac{2a}{L}\right)H\left(\frac{i\pi}{2}\right)c_i + \beta_1\left[C_i^{(1)} + D_i^{(1)}\right]\lambda_i^{(1)}$$

$$(i = 1,3,5, \ldots) \qquad \text{(A21)}$$

and

$$\sum_{j=0,1,2}^{\infty} (-1)^j a_{ij}^{(q)} = -\beta_q B_i^{(q)}\lambda_i^{(q-1)} + \beta_q\left[C_i^{(q)} + D_i^{(q)}\right]\lambda_i^{(q)}$$

$$(i = 1,3,5, \ldots)(q = 2,3, \ldots r) \qquad \text{(A22)}$$

In order to obtain equations in the form of (A21) and (A22) it should be noted that

$$\left[2R_i^{(q)}C_i^{(q)} - 1\right]\sum_{j=0,1,2}^{\infty}\frac{(-1)^j}{K_{ij}^{(q)}} = -B_i^{(q)}$$

$$(q = 1,2,3, \ldots r) \qquad \text{(A23)}$$

and

$$\left[\sum_{j=0,1,2}^{\infty}\frac{1}{K_{ij}^{(q)}} - 2R_i^{(q)}B_i^{(q)}\sum_{j=0,1,2}^{\infty}\frac{(-1)^j}{K_{ij}^{(q)}}\right] = C_i^{(q)} + D_i^{(q)}$$

$$(q = 1,2,3, \ldots r) \qquad \text{(A24)}$$

where

$$D_i^{(q)} = \frac{2R_i^{(q)}\left\{\left[\sum_{j=0,1,2}^{\infty} \frac{1}{K_{ij}^{(q)}}\right]^2 - \left[\sum_{j=0,1,2}^{\infty} \frac{(-1)^j}{K_{ij}^{(q)}}\right]^2\right\}}{1 + 2R_i^{(q)} \sum_{j=0,1,2}^{\infty} \frac{1}{K_{ij}^{(q)}}}$$

$$(q = 1,2,3, \ldots r) \qquad \text{(A25)}$$

Substituting equation (A15) into equation (A12) results in

$$\left[k_i - 2H\left(\frac{i\pi}{2}\right)^2 C_i^{(1)}\right]\frac{c_i}{\beta_1} + \frac{L}{a}\left(\frac{i\pi}{2}\right)B_i^{(1)}\lambda_i^{(1)} = 2k_B{}^2k_S{}^2\left(\frac{2}{i\pi}\right)(-1)^{\frac{i-1}{2}} \frac{C}{\beta_1}$$

$$(i = 1,3,5, \ldots) \qquad \text{(A26)}$$

and substituting equations (A21) and (A16) into equation (A8), when $p = 1$, gives

$$-B_i^{(1)}\left(\frac{2a}{L}\right)H\left(\frac{i\pi}{2}\right) \frac{c_i}{\beta_1} + \left[C_i^{(1)} + D_i^{(1)} + \frac{b_2}{b_1} C_i^{(2)}\right]\lambda_i^{(1)} -$$

$$\frac{b_2}{b_1} B_i^{(2)}\lambda_i^{(2)}\left(1 - \delta_{(r+1),2}\right) = 0$$

$$(i = 1,3,5, \ldots) \qquad \text{(A27)}$$

Substituting equations (A16) and (A22) into equation (A8) yields

$$-B_i^{(p)}\lambda_i^{(p-1)} + \left[C_i^{(p)} + D_i^{(p)} + \frac{b_{p+1}}{b_p} C_i^{(p+1)}\right]\lambda_i^{(p)} -$$

$$\frac{b_{p+1}}{b_p} B_i^{(p+1)}\lambda_i^{(p+1)}\left(1 - \delta_{(r+1),(p+1)}\right) = 0$$

$$(i = 1,3,5, \ldots)(p = 2,3, \ldots r) \qquad \text{(A28)}$$

By Cramer's rule, equations (A26) to (A28) may be solved for c_i/β_1 in terms of C/β_1; if in the resulting expression the determinants in the numerator and denominator are each expanded by the first column and if in the resulting denominator the cofactor of $-B_i^{(1)}\left(\frac{2a}{L}\right)H\left(\frac{i\pi}{2}\right)$ is also expanded by the first column, the following equation results:

$$\frac{c_i}{\beta_1} = \frac{2k_B{}^2k_S{}^2\left(\frac{2}{i\pi}\right)(-1)^{\frac{i-1}{2}}\frac{C}{\beta_1}S_i^{(1)}}{\left[k_i - 2H\left(\frac{i\pi}{2}\right)^2 C_i^{(1)}\right]S_i^{(1)} + 2H\left(\frac{i\pi}{2}\right)^2\left(B_i^{(1)}\right)^2 S_i^{(2)}} \qquad (i = 1,3,5, \ldots) \qquad \text{(A29)}$$

where

$$S_i^{(p)} = \begin{vmatrix} C_i^{(p)} + D_i^{(p)} + \frac{b_{p+1}}{b_p}C_i^{(p+1)} & \frac{-b_{p+1}}{b_p}B_i^{(p+1)} & \cdot & \cdots & \cdot & \cdot & \cdot \\ -B_i^{(p+1)} & C_i^{(p+1)} + D_i^{(p+1)} + \frac{b_{p+2}}{b_{p+1}}C_i^{(p+2)} & \frac{-b_{p+2}}{b_{p+1}}B_i^{(p+2)} & \cdots & \cdot & \cdot & \cdot \\ \cdot & -B_i^{(p+2)} & C_i^{(p+2)} + D_i^{(p+2)} + \frac{b_{p+3}}{b_{p+2}}C_i^{(p+3)} & \cdots & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdots & C_i^{(r-2)} + D_i^{(r-2)} + \frac{b_{r-1}}{b_{r-2}}C_i^{(r-1)} & \frac{-b_{r-1}}{b_{r-2}}B_i^{(r-1)} & \cdot \\ \cdot & \cdot & \cdot & \cdots & -B_i^{(r-1)} & C_i^{(r-1)} + D_i^{(r-1)} + \frac{b_r}{b_{r-1}}C_i^{(r)} & \frac{-b_r}{b_{r-1}}B_i^{(r)} \\ \cdot & \cdot & \cdot & \cdots & \cdot & -B_i^{(r)} & C_i^{(r)} + D_i^{(r)} + \frac{b_{r+1}}{b_r}C_i^{(r+1)} \end{vmatrix}$$

$$(p = 1,2,3, \ldots r) \qquad \text{(A30)}$$

By expanding the determinant of (A30) by the first column the following recurrence relation may be obtained:

$$S_i^{(p)} = \left[C_i^{(p)} + D_i^{(p)} + \frac{b_{p+1}}{b_p} C_i^{(p+1)}\right] S_i^{(p+1)} - \frac{b_{p+1}}{b_p}\left[B_i^{(p+1)}\right]^2 S_i^{(p+2)}$$

$$(p = 1,2,3, \ldots r - 1) \qquad \text{(A31)}$$

where

$$S_i^{(r)} = \left[C_i^{(r)} + D_i^{(r)} + \frac{b_{r+1}}{b_r} C_i^{(r+1)}\right]$$

and

$$S_i^{(r+1)} = 1$$

Now, by dividing the numerator and denominator of the right-hand member of equation (A29) by $S_i^{(1)}$ and then using equation (A31), the following equation results:

$$c_i = \frac{2k_B^2 k_S^2 \left(\frac{2}{i\pi}\right)(-1)^{\frac{i-1}{2}} C}{\left[k_i - 2H\left(\frac{i\pi}{2}\right)^2 C_i^{(1)}\right] + 2H\left(\frac{i\pi}{2}\right)^2 \dfrac{\left(B_i^{(1)}\right)^2 S_i^{(2)}}{\left[C_i^{(1)} + D_i^{(1)} + \frac{b_2}{b_1} C_i^{(2)}\right] S_i^{(2)} - \frac{b_2}{b_1}\left(B_i^{(2)}\right)^2 S_i^{(3)}}}$$

$$(i = 1,3,5, \ldots) \qquad \text{(A32)}$$

Repetition of this procedure ultimately yields the continued fraction

$$a_i = \frac{2k_B^2 k_s^2\left(\frac{2}{i\pi}\right)(-1)^{\frac{i-1}{2}} c}{k_i - 2H\left(\frac{i\pi}{2}\right)^2\left[C_i^{(1)} - \cfrac{\left(B_i^{(1)}\right)^2}{C_i^{(1)} + D_i^{(1)} + \frac{b_2}{b_1}\left[C_i^{(2)} - \cfrac{\left(B_i^{(2)}\right)^2}{C_i^{(2)} + D_i^{(2)} + \frac{b_3}{b_2}\left[C_i^{(3)} - \cfrac{\left(B_i^{(3)}\right)^2}{C_i^{(3)} + D_i^{(3)} + \cdots + \frac{b_{r-1}}{b_{r-2}}\left[C_i^{(r-1)} - \cfrac{\left(B_i^{(r-1)}\right)^2}{C_i^{(r-1)} + D_i^{(r-1)} + \frac{b_r}{b_{r-1}}\left[C_i^{(r)} - \cfrac{\left(B_i^{(r)}\right)^2}{C_i^{(r)} + D_i^{(r)} + \frac{b_{r+1}}{b_r} C_i^{(r+1)}}\right]}\right]}\right]}\right]}\right]}$$

$$(i = 1,3,5, \ldots) \qquad \text{(A33)}$$

Noting the definition of H and then defining $N_i^{(p)}$ by the equations

$$\frac{1}{2N_i^{(p)}} = \frac{b_p}{b_{p-1}}\left[C_i^{(p)} - \frac{\left(B_i^{(p)}\right)^2}{C_i^{(p)} + D_i^{(p)} + \frac{1}{2N_i^{(p+1)}}}\right]$$

$$(p = 1,2, \ldots r) \qquad \text{(A34)}$$

and

$$\frac{1}{2N_i^{(r+1)}} = \frac{b_{r+1}}{b_r} C_i^{(r+1)} \qquad \text{(A35)}$$

where

$$b_0 = a$$

permits equation (A33) to be written as

$$c_i = \frac{2k_B{}^2 k_S{}^2 \left(\frac{2}{i\pi}\right)(-1)^{\frac{i-1}{2}} C}{k_i - \frac{\left(\frac{i\pi}{2}\right)^2 \frac{t_W}{t_C}}{N_i{}^{(1)}}} \qquad (i = 1,3,5, \ldots) \qquad \text{(A36)}$$

Substituting equation (A36) into equation (A13) yields the frequency equation

$$k_B{}^2 \left\{ 1 + 2k_B{}^2 k_S{}^2 \sum_{m=1,3,5}^{\infty} \frac{1}{\left(\frac{m\pi}{2}\right)^2 \left[k_m - \frac{\left(\frac{m\pi}{2}\right)^2 \frac{t_W}{t_C}}{N_m{}^{(1)}} \right]} \right\} = 0 \qquad \text{(A37)}$$

The following closed forms for the summations involved in $B_m{}^{(p)}$, $C_m{}^{(p)}$, and $D_m{}^{(p)}$ may be obtained by methods similar to those given in reference 4:

$$\sum_{j=0,1,2}^{\infty} \frac{1}{K_{mj}{}^{(p)}} = \frac{1}{2\left(\frac{m\pi}{2}\right)P_p \tanh\left(\frac{m\pi}{2}\right)P_p} \qquad (p = 1,2, \ldots r + 1) \qquad \text{(A38)}$$

$$\sum_{j=0,1,2}^{\infty} \frac{(-1)^j}{K_{mj}{}^{(p)}} = \frac{1}{2\left(\frac{m\pi}{2}\right)P_p \sinh\left(\frac{m\pi}{2}\right)P_p} \qquad (p = 1,2, \ldots r + 1) \qquad \text{(A39)}$$

By use of equations (A38) and (A39), equations (A17), (A18), and (A25) may now be written as

$$B_m{}^{(p)} = \frac{\operatorname{sech}\left(\frac{m\pi}{2}\right)P_p}{2\left[\left(\frac{m\pi}{2}\right)P_p \tanh\left(\frac{m\pi}{2}\right)P_p + R_m{}^{(p)}\right]} \qquad (p = 1,2, \ldots r) \qquad \text{(A40)}$$

$$C_m{}^{(p)} = \frac{1}{2\left[\left(1 - \theta_N \delta_{(r+1),p}\right)\left(\frac{m\pi}{2}\right)P_p \tanh\left(\frac{m\pi}{2}\right)P_p + R_m{}^{(p)}\right]}$$

$$(p = 1,2, \ldots r + 1) \qquad \text{(A41)}$$

$$D_m^{(p)} = \frac{R_m^{(p)}}{2\left(\frac{m\pi}{2}\right)P_p}\left[\frac{\tanh\left(\frac{m\pi}{2}\right)P_p}{\left(\frac{m\pi}{2}\right)P_p \tanh\left(\frac{m\pi}{2}\right)P_p + R_m^{(p)}}\right]$$

$$(p = 1,2,3, \ldots r) \qquad \text{(A42)}$$

From equations (A34), (A35), (A40), (A41), and (A42) the following expressions for $N_m^{(p)}$ and $N_m^{(r+1)}$ may be obtained:

$$N_m^{(p)} = \frac{b_{p-1}}{b_p}\left[\left(\frac{m\pi}{2}\right)P_p \tanh\left(\frac{m\pi}{2}\right)P_p + R_m^{(p)} + \frac{\operatorname{sech}^2\left(\frac{m\pi}{2}\right)P_p}{\frac{\tanh\left(\frac{m\pi}{2}\right)P_p}{\left(\frac{m\pi}{2}\right)P_p} + \frac{1}{N_m^{(p+1)}}}\right]$$

$$(p = 1,2, \ldots r)(b_0 = a) \qquad \text{(A43)}$$

$$N_m^{(r+1)} = \frac{b_r}{b_{r+1}}\left[\left(1 - \theta_N\right)\left(\frac{m\pi}{2}\right)P_{r+1} \tanh\left(\frac{m\pi}{2}\right)P_{r+1} + R_m^{(r+1)}\right]$$

$$(r \geqq 1) \qquad \text{(A44)}$$

The rate of convergence of the series of frequency equations given in equation (A37) is increased by substracting the expression

$$2k_B^4 k_S^2 \sum_{m=1,3,5}^{\infty} \frac{1}{\left(\frac{m\pi}{2}\right)^2 k_m}$$

and adding the equivalent closed-form expression

$$k_B^2\left(\frac{\tan k_B k_S}{k_B k_S} - 1\right)$$

The resulting equation is

$$k_B^2\left[\tan k_B k_S + 2k_B^3 k_S^3 \sum_{m=1,3,5}^{\infty} \frac{1}{k_m^2 \frac{t_C}{t_W} N_m^{(1)} - k_m\left(\frac{m\pi}{2}\right)^2}\right] = 0 \qquad \text{(A45)}$$

where $N_m^{(1)}$ is determined by equations (A43) and (A44). It should be noted that equation (A45) will hold for N stringers where $N \neq 0$.

For the case when $N = 0$, handling equations (A12), (A13), and (A10) when $\lambda_i^{(1)} = 0$ in a manner similar to that used for the case of N stringers yields equation (A45) except that

$$N_m^{(1)} = \frac{a}{b_1}\left[P_1\left(\frac{m\pi}{2}\right) \tanh P_1\left(\frac{m\pi}{2}\right) + R_m^{(1)}\right] \tag{A46}$$

The values P_1 and $R_i^{(1)}$ are given in equations (A14b) and (A14c), respectively; the value of b_1 is, of course, b.

In order that equation (A45) together with (A46) may apply to the case for $N = \infty$, P_1 may be redefined by

$$P_1 = \frac{Eb^2 t_C'}{GL^2 t_C} \tag{A47}$$

Antisymmetrical Transverse Modes of a Free-Free Multistringer Beam

Appropriate trigonometric series for the antisymmetrical transverse modes of a free-free beam are

$$u_p(x,y_p) = \sum_{m=0,2,4}^{\infty} \sum_{n=0,1,2}^{\infty} a_{mn}^{(p)} \cos \frac{m\pi x}{2L} \cos \frac{n\pi y_p}{b_p}$$

$$(p = 1,2, \ldots r + 1) \tag{A48}$$

$$w(x) = Cx + \sum_{m=2,4,6}^{\infty} c_m \sin \frac{m\pi x}{2L} \tag{A49}$$

As in the case for the symmetrical modes the choice of the particular series was guided by the orthogonality required for simplification of the expressions in the strain-energy equation; the term Cx was added in order to allow sufficient freedom of $w(\pm L)$.

Treatment similar to that accorded the case of the symmetrical modes yields the equation

$$k_B{}^2\left[1 - k_B k_S \cot k_B k_S + 2k_B{}^4 k_S{}^4 \sum_{m=2,4,6}^{\infty} \frac{1}{k_m{}^2 \frac{t_C}{t_W} N_m{}^{(1)} - k_m\left(\frac{m\pi}{2}\right)^2}\right] = 0 \qquad \text{(A50)}$$

where k_B, k_S, and k_m are given in equations (A14f), (A14g), and (A14j), respectively, and $N_m{}^{(1)}$ is given by equations (A43) and (A44). As before, this equation holds for N stringers, where $N \neq 0$. When $N = 0$, equation (A50) may be used, where $N_m{}^{(1)}$ is defined by (A46); for $N = \infty$, the same equations are used with the definition of P_1 as given in .. respectively; the value of b_1 is, of course, b.

In order that equation (A45) together with (A46) may apply to the case for $N = \infty$, P_1 may be redefined by

$$P_1 = \frac{Eb^2 t_C{}'}{GL^2 t_C} \qquad \text{(A47)}$$

Antisymmetrical Transverse Modes of a Free-Free Multistringer Beam

Appropriate trigonometric series for the antisymmetrical transverse modes of a free-free beam are

$$u_p\left(x,y_p\right) = \sum_{m=0,2,4}^{\infty} \sum_{n=0,1,2}^{\infty} a_{mn}{}^{(p)} \cos \frac{m\pi x}{2L} \cos \frac{n\pi y_p}{b_p}$$

$$(p = 1,2, \ldots r + 1) \qquad \text{(A48)}$$

$$w(x) = Cx + \sum_{m=2,4,6}^{\infty} c_m \sin \frac{m\pi x}{2L} \qquad \text{(A49)}$$

As in the case for the symmetrical modes the choice of the particular series was guided by the orthogonality required for simplification of the expressions in the strain-energy equation; the term Cx was added in order to allow sufficient freedom of $w(\pm L)$.

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1. Davenport, William W., and Kruszewski, Edwin T.: A Substitute-Stringer Approach for Including Shear-Lag Effects in Box-Beam Vibrations. NACA TN 3158, 1954.

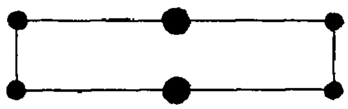
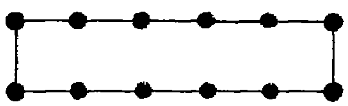
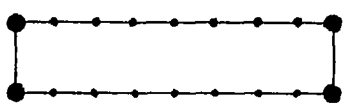
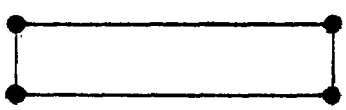
2. Budiansky, Bernard, and Kruszewski, Edwin T.: Transverse Vibrations of Hollow Thin-Walled Cylindrical Beams. NACA Rep. 1129, 1953. (Supersedes NACA TN 2682.)

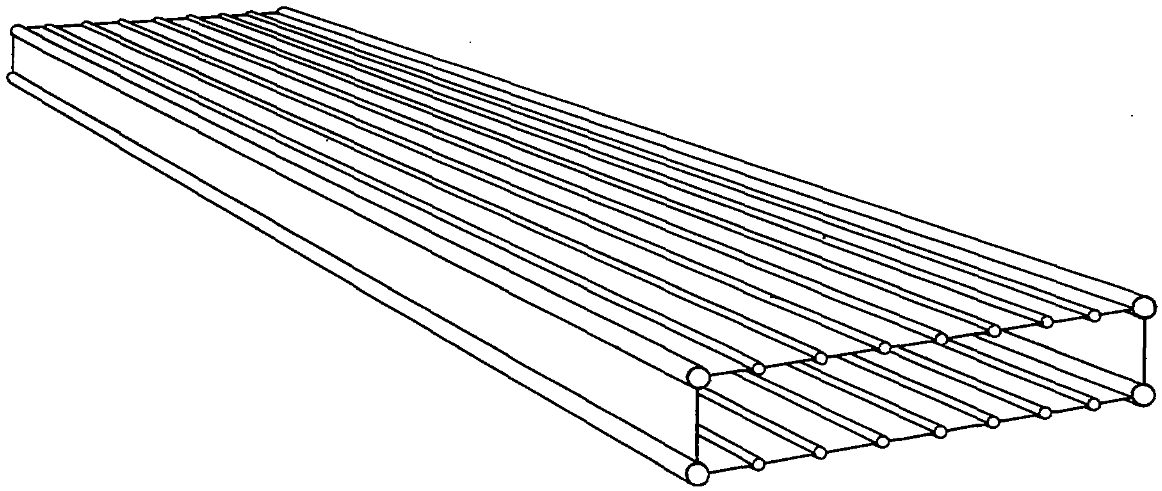
3. Kruszewski, Edwin T.: Effect of Transverse Shear and Rotary Inertia on the Natural Frequency of a Uniform Beam. NACA TN 1909, 1949.

4. Adams, Edwin P., and Hippisley, R. L.: Smithsonian Mathematical Formulae and Tables of Elliptic Functions. Second reprint, Smithsonian Misc. Coll., vol. 74, no. 1, 1947, p. 129.

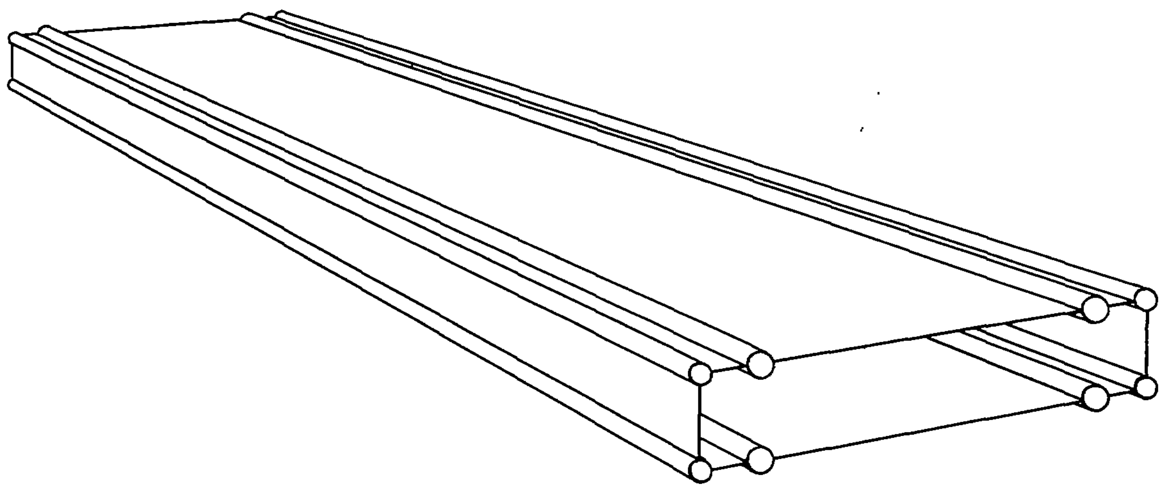
TABLE I

EFFECT OF STRINGER DISTRIBUTION ON FREQUENCY

Case	Solution	k_B		
		1st symm. mode	2d symm. mode	3d symm. mode
N = 1 	Elementary + transverse shear	5.42	24.5	48.3
	Substitute-stringer, $b_S/b_C = 0.5$	5.16	20.1	37.8
	Multistringer	5.07	19.7	38.1
N = 4 	Elementary + transverse shear	5.42	24.5	48.3
	Substitute-stringer, $b_S/b_C = 0.5$	5.22	20.8	39.1
	Multistringer	5.15	20.2	38.8
N = 7 	Elementary + transverse shear	5.42	24.5	48.3
	Substitute-stringer, $b_S/b_C = 0.5$	5.22	20.9	39.2
	Multistringer	5.16	20.4	39.1
N = ∞ 	Elementary + transverse shear	5.42	24.5	48.3
	Substitute-stringer, $b_S/b_C = 0.5$	5.23	21.0	39.5
	Multistringer	5.19	20.8	39.9

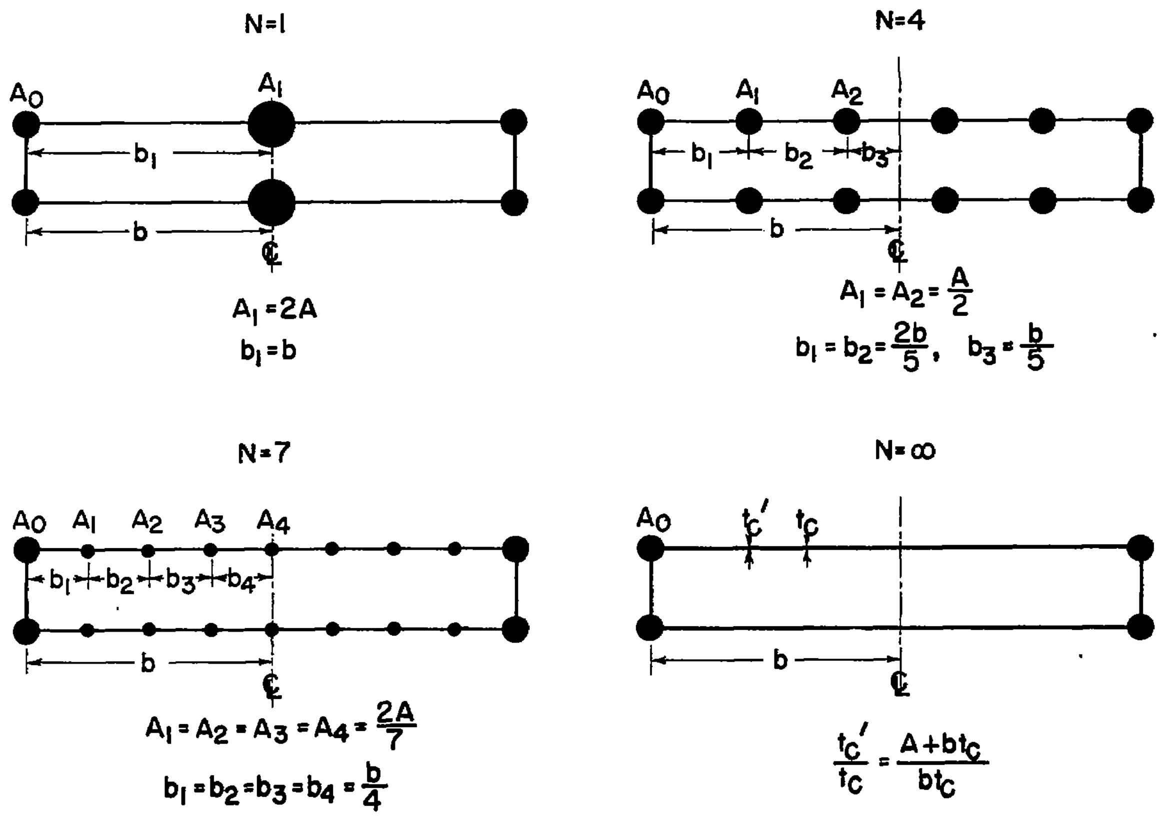


(a) Multistringer box beam.

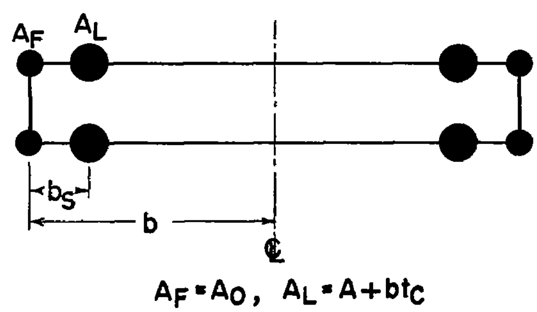


(b) Substitute-stringer structure.

Figure 1.- Multistringer box beam and its substitute-stringer structure.



(a) Multistringer beams.



(b) Substitute-stringer beam.

Figure 2.- Cross sections of multistringer beams and substitute-stringer beam.

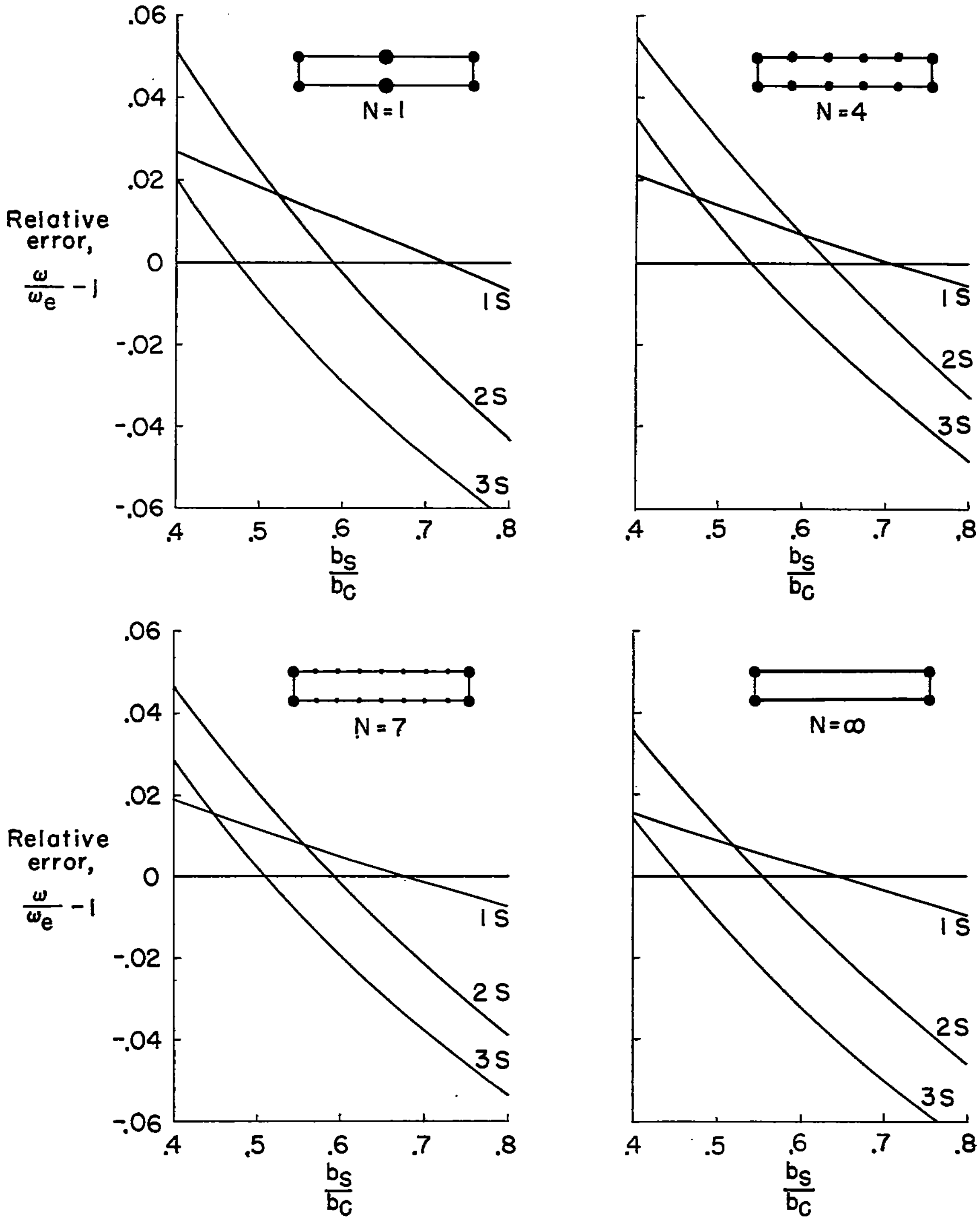


Figure 3.- Effect of substitute-stringer location on the accuracy of the substitute-stringer approach for box beams with different stringer distributions. The labels 1S, 2S, and 3S denote symmetrical modes.

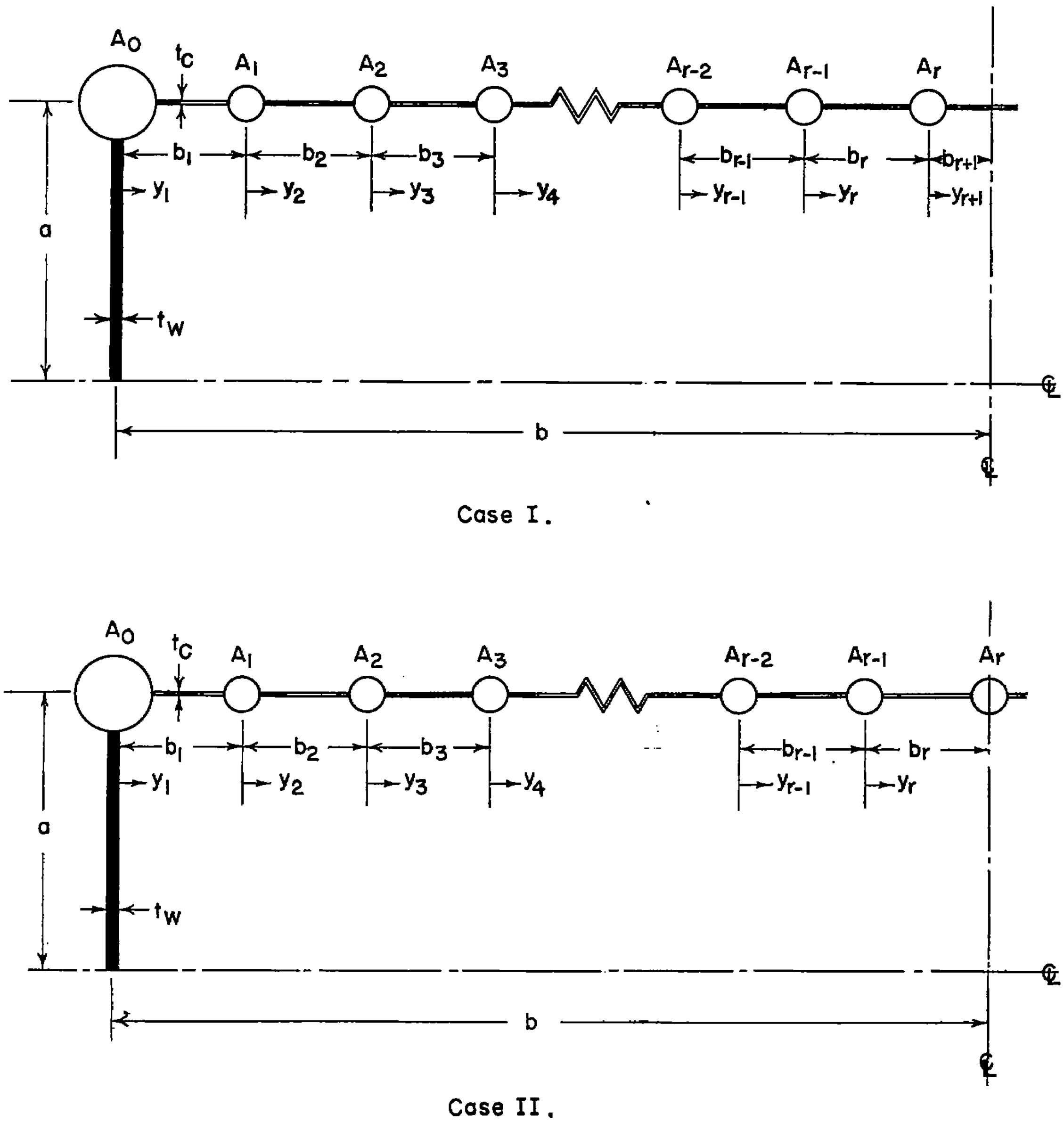


Figure 4.- Cross section of doubly symmetrical multistringer box beam for either even or odd number of stringers in a cover.